Semifields, and their relations to coding theory and cryptography

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What is a semifield?

Definition

A (finite) semifield $\mathbb{S} = (S, +, \circ)$ is a finite set S equipped with two operations $(+, \circ)$ satisfying the following axioms.

(S1) $(S, +)$ is a group.

(S2) For all $x, y, z \in S$,

- $x \circ (y + z) = x \circ y + x \circ z$,
- $(x + y) \circ z = x \circ z + y \circ z$.

(S3) For all $x, y \in S$, $x \circ y = 0$ implies $x = 0$ or $y = 0$.

(S4) There exists $\epsilon \in S$ such that $x \circ \epsilon = x = \epsilon \circ x$.

If (S4) does not hold, we call $\mathbb{S}$ a pre-semifield.

Basic properties

Every pre-semifield can easily be turned into a semifield using *Kaplansky's trick*.

The additive group of a semifield $(\mathbb{S}, +, \circ)$ is always an elementary abelian p-group.

We can thus identify the additive group of a semifield $\mathbb{S}$ with the additive group of the finite field $\mathbb{F}_{p^n}$.

To define a semifield, it then suffices to specify the "multiplication" $\circ$.

An example of a family of semifields

Example (Twisted fields, Albert, 1961)

Let $K = \mathbb{F}_{p^n}$, $n > 2$, and define $\circ\colon K \to K$ via

$$x \circ y = xy - ax^q y^r,$$

where $a \notin \mathbb{F}_{p^n}^{q-1} \cdot \mathbb{F}_{p^n}^{r-1}$ and q, r are powers of p. Then $\mathbb{S} = (K, +, \circ)$ is a semifield.

The twisted fields are *commutative* if we choose $q = r$.

Why do we care? - Algebra

Theorem (Wedderburn's theorem)

Every finite domain is a field.

In other words: There is no difference between finite domains, finite division rings, and finite fields.

Semifields are the algebraic structures "closest" to finite fields.

Commutative semifields are especially interesting.

Why do we care? - Algebra

Theorem

There exists (up to isomorphism) exactly one finite field of size p^n with p prime, $n \geq 1$.

There are more semifields, but they are still *hard to find* (esp. *commutative semifields*)!

Theorem (Menichetti, 1996)

All semifields of size p^n with p, n prime and p large enough are (equivalent to) finite fields or twisted fields.

Why do we care? - Geometry

Semifields can be used to construct *finite projective planes*.

Projective planes can be divided up into 6 groups based on their symmetries. One of them is derived from semifields.

Theorem. Let $\pi = (\mathfrak{P}, \mathfrak{L}, I)$ be a projective plane. Then exactly one of the following seven statements is true.

Lenz type	Lenz figure	Coordinatizing ternary field
I	$\mathfrak{E}_\pi = \emptyset$	Ternary fields
II	$\mathfrak{E}_\pi = \{(a, z)\}$	Cartesian groups
III	There exist a point z and a line l with $z \notin l$ such that $\mathfrak{E}_\pi = \{(p \vee z, p) : p \in l\}$	Special Cartesian groups
IVa	There exists a line a such that $\mathfrak{E}_\pi = \{a\} \times a$	Quasifields
IVb	There exists a point z such that $\mathfrak{E}_\pi = \mathfrak{L}_z \times \{z\}$	Dual of **IVa**
V	There exist a line a and a point z on a such that $\mathfrak{E}_\pi = \{a\} \times a \sqcup \mathfrak{L}_z \times \{z\}$	Semifields
VII	$\mathfrak{E}_\pi = \{(a, z) \in \mathfrak{L} \times \mathfrak{P} : z \in a\}$	Alternative fields

Constructing semifields $\Leftrightarrow$ Constructing projective planes of Lenz type V.

Why do we care? - Coding theory

Semifields can be used to construct *rank-metric codes*.

Definition (Rank-metric code)

Let $M_{n,m}(\mathbb{F}_q)$ the set of $n \times m$-matrices over $\mathbb{F}_q$. A linear *rank-metric code* is a subspace $\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ with minimum distance

$$d = \min_{X,Y \in \mathcal{C}, X \neq Y} \mathrm{rk}(X - Y).$$

The distance used here is the *rank metric*. Main problems:

- Find codes that are "optimal" (achieve maximal $|\mathcal{C}|$ with fixed n, m, d)
- Find for a given $X \in M_{n,m}(\mathbb{F}_q)$ the element in $\mathcal{C}$ closest to X. (decoding problem)

Constructions of rank-metric codes

What is a *good* rank-metric code?

Theorem (Singleton-like bound, Delsarte 1978)

Suppose $\mathcal{C} \leq M_{n,m}(\mathbb{F}_q)$ with minimum distance d. Then

$$|\mathcal{C}| \leq q^{n(m-d+1)}.$$

Rank-metric codes satisfying the bound with equality are called maximum rank distance (MRD) code.

There are few constructions of MRD codes and even less have efficient decoding algorithms.

Most constructions of MRD codes are related to *semifields*.

Connecting semifields and MRD codes

We have a simple key connection.

Theorem

Let $\mathbb{S} = (\mathbb{F}_q^n, +, \circ)$ be a semifield. Then the set of left-multiplications

$$L_x(y) = x \circ y,\ \mathcal{C} = \{L_x : x \in \mathbb{F}_q^n\}$$

defines a linear MRD code with parameters $d = m = n$.

The MRD codes constructed by semifields are *square, full rank* MRD codes.

Note: The distributivity law $x \circ (y + z) = x \circ y + x \circ z$ implies that L_x is a linear mapping.

Connecting semifields and MRD codes

Even more:

Theorem (de la Cruz, Kiermaier, Wassermann, Willems, 2015)

There is a 1-1 correspondence between finite semifields and linear, square full rank MRD codes.

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And even more: Almost all other known constructions of MRD codes start from a square full rank MRD code - and are thus connected to semifields.

Why do we care? - Code based Cryptography

Rank-metric codes can be used in *code based cryptography*.

Many code-based cryptosystems rely on the hardness of the decoding problem for random codes.

Decoding in the rank-metric is generally considered to be harder than in the classical setting.

Rank-metric code-based cryptography is a new and exciting alternative to classical code-based cryptography.

2 out of 7 second round NIST post-quantum cryptography candidates based on code based cryptography used rank-metric codes.

BUT: Families of good codes like MRD codes need to be constructed.

Why do we care? - Symmetric Cryptography

To resist differential attacks, a block cipher needs to be nonlinear.

Definition

A function $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ is called *perfect nonlinear* if the equation (in x)

$$F(x + a) - F(x) = b$$

has exactly one solution for any b and any non-zero a.

Theorem (Coulter, Henderson, 2007)

If p is odd and $\mathbb{S} = (\mathbb{F}_p^n, +, \circ)$ is a commutative semifield, then $F(x) = x \circ x$ is perfect nonlinear.

Constructions of new commutative semifields give new perfect nonlinear functions.

Objective

Goal

Construct new semifields.

Goal

Give bounds on the total number of semifields.

New semifields give new projective planes, new MRD codes, and new perfect nonlinear functions.

Commutative semifields are especially interesting.

Different semifields can be *equivalent*, and we want to construct new examples *up to equivalence*.

Definition (Isotopy)

Two semifields $\mathbb{S}_1 = (\mathbb{F}_p^n, +, \circ_1)$ and $\mathbb{S}_2 = (\mathbb{F}_p^n, +, \circ_2)$ are *isotopic* if there exist $\mathbb{F}_p$-linear bijections L, M and N of $\mathbb{F}_{p^n}$ satisfying

$$N(x \circ_1 y) = L(x) \circ_2 M(y).$$

Such a triple $\gamma = (N, L, M)$ is called an *isotopism* between $\mathbb{S}_1$ and $\mathbb{S}_2$.

Definition (Autotopism group)

The autotopism group $\mathrm{Aut}(\mathbb{S})$ of a semifield $\mathbb{S} = (\mathbb{F}_p^n, +, \circ)$ is defined by

$$\mathrm{Aut}(\mathbb{S}) = \{(N, L, M) \in \mathrm{GL}(\mathbb{F}_{p^n})^3 \colon N(x \circ y) = L(x) \circ M(y)\}.$$

Two semifields are isotopic iff the associated planes are isomorphic iff the associated rank-metric codes are equivalent.

Bivariate semifields

Example (Dickson, 1905)

Let $K = \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ with p odd and define $\circ\colon K \times K \to K$ via

$$(x, y) \circ (u, v) = (xu + a(yv)^q, xv + yu),$$

where a is a non-square in $\mathbb{F}_{p^m}$ and q is a power of p. Then $\mathbb{S} = (K, +, \circ)$ is a (commutative) semifield.

This is a *bivariate construction*.

There are many bivariate constructions!

The known commutative semifields of size p^n, p odd

Until 2022:

Family	**Count**	Proven in	Bivariate?
The finite field	1	trivial	$\approx$ Yes
Dickson	$\approx n/4$	1905	Yes
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Budaghyan-Helleseth	$\approx n/2$	2009	Yes
Bierbrauer_3	$\approx n/2$	2010	No
Bierbrauer_4	$\approx n/2$	2010	No
Zhou-Pott	$\approx n^2$	2013	Yes

The known commutative semifields of size p^n, p odd

Question

How many semifields are there?

Open problem!

The main problem in connection with commutative semifields of order p^n is the following:

Problem 8.19 *Decide whether the number of nonisotopic (commutative) semifield planes can be bounded by a polynomial in n.*

Pott, A.: Almost perfect and planar functions, *Designs, Codes, Cryptography*, 2016.

Bivariate constructions

We are interested in special bivariate constructions where $K = \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ and

$$(x, y) \circ (u, v) = (f(x, y, u, v), g(x, y, u, v)),$$

and f, g are homogeneous of degree $q + 1$ (resp. $r + 1$) where q, r are powers of p.

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Example (Göloglu, K., 2022)

Let $K = \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$, m even and set

$$(x, y) \circ (u, v) = (x^q u + x u^q + b(y^q v + y v^q), x^r v + y u^r + a/b(y v^r + y^r v)),$$

where p odd, $q = p^k$, $r = p^{k+m/2}$, $b \in \mathbb{F}_{p^m}$ is a non-square, $a \in \mathbb{F}^*_{p^{m/2}}$, $m/\gcd(k, m)$ is odd.

Why this structure?

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and f, g are homogeneous of degree $q + 1$ (resp. $r + 1$) where q, r are powers of p.

These semifields have some nice autotopisms! Namely, if $L = M = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ then

$$L(x, y) \circ M(u, v) = (a^{q+1} f(x, y, u, v), a^{r+1} g(x, y, u, v)),$$

so (N, L, M) with $N = \begin{pmatrix} a^{q+1} & 0 \\ 0 & a^{r+1} \end{pmatrix}$ is an autotopism for any $a \in \mathbb{F}_{p^m}^{\times}$.

$\Longrightarrow$ These semifields always have a cyclic subgroup in their autotopism group of order $p^m - 1$.

Why this structure?

Another reason is: It turns out many of the known bivariate semifields semifields "secretly" have this structure!

Example (Zhou-Pott, 2013)

Let $K = \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$.

$$(x, y) \circ (u, v) = (x^q u + u^q x + \alpha(y^q v + y v^q)^r, xv + yu)$$

where $q = p^k$, $r = p^l$, $\gcd(k, m)/m$ is odd, and α is a non-square in $\mathbb{F}_{p^m}$.

..is isotopic to...

$$(x, y) \circ (u, v) = (x^q u + u^q x + \alpha(y^q v + y v^q), x^r v + y u^r).$$

And many more (e.g. Dickson, Budaghyan-Helleseth....)!

What can we do with this structure?

1. Systematically search for new semifields that have this structure.

2. Use the nice subgroup in the autotopism group to answer the isotopy question!

Isotopy of semifields

Question

How can we decide if different semifields are isotopic or not? Can we count the (known) semifields up to isotopy?

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where p odd, $q = p^k$, $r = p^{k+m/2}$, $a \in \mathbb{F}^*_{p^{m/2}}$, $b \in \mathbb{F}_{p^m}$ is a non-square, $m/\gcd(k, m)$ is odd.

Which choices for q, a, b yield non-isotopic semifields? This is in general a very hard question!

Isotopy via the autotopism group

Lemma

Assume $\mathbb{S}_1, \mathbb{S}_2$ *are isotopic semifields of order* p^n. *Then* $\mathrm{Aut}(\mathbb{S}_1)$ *and* $\mathrm{Aut}(\mathbb{S}_2)$ *are conjugate in* $\mathrm{GL}(\mathbb{F}_{p^n})^3$.

Problem: Determining the autotopism group is also very hard!

There is sometimes a way to use the lemma without knowing the autotopism group - if one can identify a large and nice subgroup first.

Recall our bivariate semifields have a cyclic subgroup of order $p^m - 1$ in the autotopism group!

Show that two bivariate semifields $\mathbb{S}_1, \mathbb{S}_2$ are not isotopic - in five simple steps!

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- Let $H_1 \leq \mathrm{Aut}(\mathbb{S}_1)$, $H_2 \leq \mathrm{Aut}(\mathbb{S}_2)$ with $|H_1| = |H_2| = p^m - 1$ be the nice cyclic autotopism subgroups.

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- Choose a suitable prime p' and Sylow p'-groups $S_1 \leq H_1$, $S_2 \leq H_2$.

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- Choose a suitable prime p' and Sylow p'-groups $S_1 \leq H_1$, $S_2 \leq H_2$.
- Prove that S_1, S_2 are also Sylow p'-groups of $\mathrm{Aut}(\mathbb{S}_1), \mathrm{Aut}(\mathbb{S}_2)$ (key step!)

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- If $\gamma^{-1}\,\mathrm{Aut}(\mathbb{S}_1)\gamma = \mathrm{Aut}(\mathbb{S}_2)$ then $\gamma^{-1}S_1\gamma$ is a Sylow subgroup of $\mathrm{Aut}(\mathbb{S}_2)$. So $\gamma^{-1}S_1\gamma$ and S_2 are conjugate in $\mathrm{Aut}(\mathbb{S}_2)$ (by Sylow's theorem)!

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- Determine all $\delta \in \mathrm{GL}(\mathbb{F}_{p^n})^3$ such that $\delta^{-1}S_1\delta = S_2$. If those δ are not isotopisms, then $\mathbb{S}_1$, $\mathbb{S}_2$ are not isotopic.

In some sense, checking $\gamma^{-1}\,\mathrm{Aut}(\mathbb{S}_1)\gamma = \mathrm{Aut}(\mathbb{S}_2)$ is reduced to checking $\delta^{-1}S_1\delta = S_2$.

From this procedure we get the following result:

Theorem (Göloğlu, K., 2022)

If two sufficiently nice bivariate semifields defined over $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ are isotopic then there exists an isotopism $\gamma = (N, L, M) \in \Gamma L(2, \mathbb{F}_{p^m})^3$ between them.

This simplifies the isotopy question for all nice bivariate semifields.

Isotopisms of the form $\gamma = (N, L, M) \in \Gamma L(2, \mathbb{F}_{p^m})^3$ are (comparatively) easy to determine.

The known commutative semifields of size p^n, p odd

Family	**Count**	Proven in	Bivariate?
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Göloğlu-K.	$\approx p^{n/4}$	2022	Yes

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The main problem in connection with commutative semifields of order p^n is the following:

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This also yields the biggest family of commutative MRD codes, commutative semifield planes, and perfect nonlinear functions!

The known non-commutative semifields of size p^n, p odd

Non-commutative semifields:

- There are more constructions, e.g. via skew-polynomial rings (Petit, 1966), finite geometry (Jha, Johnson, 1990) or secondary constructions based on commutative semifields
- However, counting (up to isotopy) is much more difficult!
- Several families have $\approx p^{n/2}$ non-isotopic elements (Kantor 2003, Lavrauw 2013, Sheekey 2019)

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- However, counting (up to isotopy) is much more difficult!
- Several families have $\approx p^{n/2}$ non-isotopic elements (Kantor 2003, Lavrauw 2013, Sheekey 2019)
- The "square-root barrier" was broken in (Göloğlu, K. , 2023). We presented a family with $\approx p^{2n/3}$ non-isotopic semifields.

Current and future work

Final goal:

Problem (Kantor's conjecture, 2003)

Prove that the number of non-isotopic semifields of odd order $N = p^n$ is at least exponential in N.

The best current bound is $p^{2n/3}$, not even linear in N.

Interestingly, in characteristic 2 a family with exponentially many semifields has been found (Kantor and Williams, 2004).

Current and future work

New constructions:

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Multivariate semifields???: Use $\mathbb{F}_{p^{km}} \cong \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \times \cdots \times \mathbb{F}_{p^m}$.

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Current work: We actually have found some trivariate semifields and are working on a paper.

But: Adapting the group theoretical framework is not as straightforward.

Current and future work - Autotopism groups

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Determining the autotopism group of a semifield is equivalent to computing the collineation group of the associated projective plane and the automorphism group of the associated rank-metric code.

For the bivariate Knuth semifields this is an old open conjecture due to Hughes and Albert:

> Our treatment of the collineation group leaves unanswered a number of possibly interesting questions: (1) Since we only determine a sub-normal series for the group, is the group itself amenable to direct computation? (2) What is the transitive structure of the group, and more particularly, what are the transitive constituents on the line at infinity of the autotopism group $\mathfrak{G}$?

Hughes, D.R.: Collineation Groups of Non-Desarguesian Planes II. Some Division Algebras, *American Journal of Mathematics*, 1960.

Current and future work - Autotopism groups

Theorem (Göloğlu, K., 2023+)

If $\mathbb{S}$ is a bivariate Knuth semifield defined over $\mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$, we have $\mathrm{Aut}(\mathbb{S}) \leq \Gamma L(2, \mathbb{F}_{p^m})^3$.

This allows us to calculate the autotopism groups of the Knuth semifields.

Proof needs more sophisticated group theory (e.g. theory of Frobenius groups) as well as a detailed treatment of the splitting behavior of so called *projective polynomials*.

This result solves the 60 year old conjectures by Hughes and Albert.

Current and Future work - Coding theory and cryptography

Rank-metric codes from semifields:

- Constructions of MRD codes based on new semifields we found.
- Adapting decoding algorithms of existing MRD codes to new MRD codes.
- Check if cryptosystems using the new codes are resistant to known attacks on rank-metric code based cryptography (e.g. Overbeck's attack).

Shoutouts

Parts of the talk are based on:

Göloğlu, F., Kölsch, L.: An exponential bound on the number of non-isotopic commutative semifields. *Transactions of the American Mathematical Society*, 2022.

Göloğlu, F., Kölsch, L.: Counting the number of non-isotopic Taniguchi semifields. To appear in *Designs, Codes, Cryptography*, 2023.

Göloğlu, F., Kölsch, L.: Equivalences of biprojective almost perfect nonlinear functions. To appear in *Journal of Combinatorial Theory, Series A*, 2023.

Kölsch, L., Polujan, A.: Value distributions of perfect nonlinear functions, submitted, 2023.

Thank you for your attention!

Göloğlu, F., Kölsch, L.: An exponential bound on the number of non-isotopic commutative semifields. *Transactions of the American Mathematical Society*, 2022.

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